Journal of Engineering Mathematics, Vol. 6, No. 4, October 1972
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

A Theory for the Design of Thin Heat Flux Meters

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(Received February 14, 1972)

SUMMARY
Heat flux meters are analysed asymptotically for small thickness/width ratios. The gain of the meter is calculated for several examples, both two-dimensional and axisymmetric. Ideal and optimum designs for meters are suggested.

1. Introduction

Heat flux meters are devices used in micro-meteorology to measure local directional transfer of heat, e.g. vertically in soil. They usually consist of thin discs aligned with faces normal to the required direction of heat transfer. The disc contains a thermopile to measure the temperature difference across its faces.

Since such a device obviously disturbs its environment, a number of theoretical and experimental investigations have been made (e.g. Portman [5], Philip [4], and Schwerdtfeger [6]) of its calibration properties. A natural conclusion is that for minimum disturbance to the ambient heat flow, the meter should be as thin as possible. That is, if ε denotes the ratio between maximum thickness and maximum diameter, we should have $\varepsilon \ll 1$.

The theoretical portion of the work of Portman [5] and of Schwerdtfeger [6] is of a general dimensional and empirical nature, and is intended to provide calibration laws for arbitrary (non-small) ε, even to the point of remaining valid when the "meter" has degenerated into an ideal temperature probe, a slender object with $\varepsilon \gg 1$. In this wide range of ε, Schwerdtfeger [6] suggests that Philip's [4] result

$$G = \frac{\bar{K}}{\kappa + (\bar{K} - \kappa) H} \tag{1.1}$$

for the gain G (ratio between measured and ambient heat flux) is of general validity*. Here $\bar{K}$ is the mean thermal conductivity of the meter material, and κ that of the surrounding medium. The parameter H depends only on the geometry of the meter, and Schwerdtfeger suggests the empirical law.

$$\log H = -\tfrac{1}{2}[\log \varepsilon + (\log^2 \varepsilon + A)^{\frac{1}{2}}] \tag{1.2}$$

to describe its variation with thickness ε. The constant A is independent of thickness and depends only on the qualitative shape of the meter sections (rectangular, elliptic etc.).

In fact (1.1) is an exact result for oblate spheroidal meters (Philip [4]), where $H = H(\varepsilon)$ is a complicated function of ε, which reduces when ε is small to

$$H = 1 - \alpha\varepsilon , \tag{1.3}$$

with $\alpha = \pi/2$. Philip [4] suggested (1.3) as a universal law for small ε, where α is a shape parameter like A in (1.2), taking values near to $\pi/2$. However, (1.2) appears to fit experimental measurements better for large ε.

$\star$ The notation used here differs considerably from that of previous authors. In particular, our G = Philip's f = Schwerdtfeger's B_m/B_s, our ε = Philip's η = Schwerdtfeger's $1/G$, our $\bar{K}/\kappa$ = Philip's ε = Schwerdtfeger's k_m/k_s etc. Philip also uses a thickness ratio r based on square root of base area, and the parameter α in equation (1.3) differs by a factor of about 0.9 from that used by Philip in conjunction with r.

On the other hand, in the limit as the meter becomes infinitesimally thin ($\varepsilon \to 0$), both (1.2) and (1.3) predict that $H \to 1$, and hence (1.1) gives $G \to 1$. This is to be expected since an infinitesimally thin meter with $\varepsilon = 0$ does not disturb the environment at all. This is also true for arbitrary ε when $\bar{K} = \kappa$, as is clear physically and follows from (1.1).

Our concern in the present paper is with small but non-zero ε, and we show that as a first order result for small ε, Philip's [4] result (1.3) is indeed valid for arbitrary meter shapes. It should be noted that if ε is small, (1.1) and (1.3) together reduce to

$$G = 1 + \alpha\left(1 - \frac{\kappa}{\bar{K}}\right)\varepsilon\,, \tag{1.4}$$

an approximate formula with error $O(\varepsilon^2)$.

We provide here means of estimating the parameter α for general meter shapes, both for two-dimensional and axisymmetric meters. We confirm the value $\alpha = \pi/2$ for oblate spheroidal meters, but find no evidence that α remains close to $\pi/2$ for general shapes. Indeed zero or even negative values of α are possible.

The formula (1.4) has the required property that $G = 1$ when $\varepsilon = 0$ and when $\kappa = \bar{K}$. In addition, we observe that $G = 1$ when $\alpha = 0$, to the present order of approximation with respect to ε. Thus thin meters with $\alpha = 0$ are "optimum", in that they appear transparent to heat. This optimum is dependent in general on the location of the thermopile within the meter; we consider in addition "ideal" meters which have $G = 1$ for *every* possible location of the thermopile.

The above discussion relates to meters of effectively uniform internal conductivity $\bar{K}$. If $\bar{K}$ varies across the diameter of the meter (as when shielding edges are present, for example), the formula (1.4) is replaced by

$$G = 1 + \varepsilon(\alpha - \beta\kappa) \tag{1.5}$$

where α, as before depends only on meter shape, while β depends on shape and on the variable meter conductivity $\bar{K}$, but not on thickness ε or on external conductivity κ. The most important optimisation problem now is to choose $\beta = 0$, even if we are unable to make α vanish, since if $\beta = 0$ we obtain a gain of $1 + \varepsilon\alpha$ which can be calibrated once and for all independent of the (usually unknown) external conductivity κ.

2. Formulation of the General Boundary Value Problem

We consider first a very general situation as sketched in Figure 1, in which the meter is an arbitrary closed body with surface S, situated in a thermally uniform infinite medium with a uniform temperature gradient T' at a great distance from S. We choose the direction of the temperature gradient as the y-axis, so that if $T(x, y, z)$ is the temperature of the surrounding medium,

$$T \to T'y \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty\,. \tag{2.1}$$

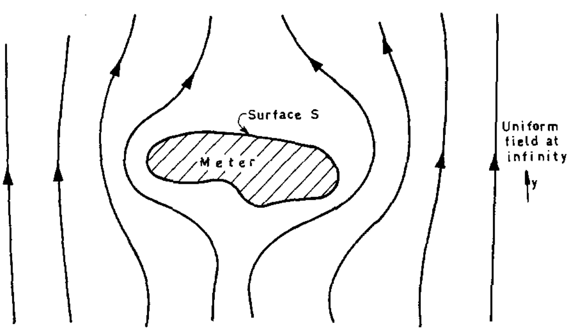

Figure 1. Sketch of general heat flow problem.

It is convenient to define a non-dimensional temperature field $\phi(x, y, z)$ such that

$$T = T'\phi \tag{2.2}$$

and

$$\phi \to y \text{ as } x^2+y^2+z^2 \to \infty\,. \tag{2.3}$$

We also suppose that the space co-ordinates have already been scaled with respect to a typical length scale, specifically the half-width of the meter in the x-direction.

In the general case we may allow for an arbitrary geometric shape of the meter by taking as the equation of its upper and lower faces respectively

$$y = f_{\pm}(x, z)\,, \quad |x| < 1\,, \tag{2.4}$$

where $f_+(x, z)$, $f_-(x, z)$ are given functions satisfying $f_+ > f_-$. The equation to be satisfied by the temperature field ϕ under steady state conditions is Laplace's equation

$$\phi_{xx}+\phi_{yy}+\phi_{zz} = 0\,, \quad y > f_+ \text{ and } y < f_-\,. \tag{2.5}$$

In addition to the boundary condition (2.3) at infinity, ϕ must satisfy boundary conditions on S which match it with the temperature field inside the meter.

If the meter has a non-uniform conductivity $K(x, y, z)$, the equation describing the steady state (non-dimensional as in (2.2)) temperature $\Phi(x, y, z)$ inside the meter is

$$\nabla \cdot K \nabla \Phi = 0\,, \quad f_- < y < f_+ \tag{2.6}$$

which reduces to Laplace's equation in the case of a uniform meter K=constant. At the boundary S we must have continuity of temperature, i.e.

$$\phi = \Phi \text{ on } S \tag{2.7}$$

or

$$\phi(x, f_{\pm}(x, z), z) = \Phi(x, f_{\pm}(x, z), z)\,, \tag{2.8}$$

and of heat flux, i.e.

$$\kappa \frac{\partial \phi}{\partial n} = K \frac{\partial \Phi}{\partial n} \text{ on } S \tag{2.9}$$

where κ is the (uniform) conductivity of the external medium, and $\partial/\partial n$ denotes differentiation normal to S.

The boundary value problem thus formulated possesses a unique solution and can be solved for any specification of S (via $f_{\pm}$) and of K. However the solution can only be written down in closed form when the meter conductivity K is constant, and even then only for idealized geometries such as ellipsoids (Philip, [4]). In any case, a good meter is bound to be "thin", in the sense of, and for the reasons given in, the following sections.

3. The Thin Meter Approximation

We now suppose that the meter is symmetrical about its centre plane $y=0$, and write

$$f_{\pm}(x, z) = \pm\varepsilon F(x, z) \tag{3.1}$$

where ε is a measure of the thickness of the meter (relative to its width, since we have non-dimensionalized), and F is a given function which remains of order unity as ε becomes small. Symmetry of the situation enables us to confine attention to the upper half space $y \geqq 0$ as indicated in Figure 2, setting

$$\phi = 0 \text{ on } y = 0 \text{ outside } S \tag{3.2}$$

and

$$\Phi = 0 \text{ on } y = 0 \text{ inside } S\,. \tag{3.3}$$

The boundary value problem formulated in section 2 may now be solved, or at least sim-

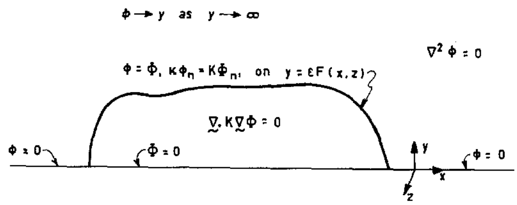

Figure 2. Boundary value problem for symmetric meters.

plified, by a perturbation expansion in the limit as $\varepsilon \to 0$. One technique (Van Dyke, [9]) for carrying out such an asymptotic expansion is to first "stretch" the y-coordinate with respect to ε, e.g. by setting $y = \varepsilon Y$. The net effect of this stretching operation on the equation (2.6) for the meter temperature Φ is that x and z derivatives of Φ may be neglected compared with y-derivatives, reducing (2.6) to

$$\frac{\partial}{\partial y}\left(K \frac{\partial \Phi}{\partial y}\right) = 0\,, \tag{3.4}$$

which simply states that the heat flux $K\Phi_y$ is uniform across the meter to this order of approximation.

Thus we may write in the limit as $\varepsilon \to 0$,

$$\Phi(x, y, z) = H(x, z) \int_0^y \frac{d\eta}{K(x, \eta, z)} \tag{3.5}$$

for some unknown function $H(x, z)$. The expression (3.5) already satisfies the anti-symmetry condition (3.3). A similar approximation to the outer temperature distribution ϕ leads simply to the conclusion that for small ε and $y = O(\varepsilon)$ we may use the Taylor series

$$\phi(x, y, z) = \phi(x, 0, z) + y\phi_y(x, 0, z) + O(\varepsilon^2)\,. \tag{3.6}$$

We now consider the effect of the boundary condition (2.9), in which to leading order in ε the normal derivative $\partial/\partial n$ may be replaced by a derivative $\partial/\partial y$ normal to the centre plane. Then (2.9), combined with (3.5) and (3.6), implies that

$$H(x, y) = \kappa\phi_y(x, 0, z)\,, \tag{3.7}$$

i.e. H is equal to the *apparent* heat flux across the plane $y = 0$ associated with the *exterior* field ϕ.

Finally, use of the boundary condition (2.8) gives

$$\phi(x, 0, z) + \varepsilon F(x, z)\phi_y(x, 0, z) = \kappa\phi_y(x, 0, z) \int_0^{\varepsilon F(x,z)} \frac{d\eta}{K(x, \eta, z)}$$

i.e.
$$\phi(x, 0, z) = \Gamma(x, z)\phi_y(x, 0, z) \tag{3.8}$$

where

$$\Gamma(x, z) = \kappa \int_0^{\varepsilon F(x,z)} \frac{d\eta}{K(x, \eta, z)} - \varepsilon F(x, y) \tag{3.9}$$

$$= \left(\frac{\kappa}{\bar{K}(x, z)} - 1\right) \varepsilon F(x, z)\,. \tag{3.10}$$

In equation (3.10)

$$\bar{K}(x, z) = \frac{\varepsilon F(x, z)}{\displaystyle\int_0^{\varepsilon F(x,z)} \frac{d\eta}{K(x, \eta, z)}} \tag{3.11}$$

is the (series) mean conductivity across the meter at the location (x, z). In the special case $K = \text{constant}$, $\bar{K} = K$; somewhat more generally, a meter consisting of layers each having uniform conductivity would lead to the well-known result

$$\bar{K} = \frac{\text{total thickness}}{\sum \dfrac{\text{layer thickness}}{\text{layer conductivity}}} \tag{3.12}$$

(e.g. Portman [5]).

Equation (3.8) is of fundamental importance in this theory, and constitutes a "mixed" boundary condition to be applied on the centre plane $y=0$ instead of on the actual surface of the meter. The boundary value problem for ϕ is hence much simplified. The total information about the meter has been condensed into the form of the function $\Gamma(x, z)$, defined by (3.10) and supposed known, which relates the temperature ϕ and its normal derivative ϕ_y on $y=0$. Γ may be described as the "impedance" of the meter.

In fact, however, (3.8) is inconsistent with respect to ε and still unnecessarily complicated as it stands. This is because, as indicated by (3.10), Γ is itself small when ε is small, and we may simplify further. To accomplish this, we first subtract off the uniform field at infinity, writing

$$\phi(x, y, z) = y + \phi'(x, y, z)\,, \tag{3.13}$$

where ϕ' still satisfies Laplace's equation and vanishes on $y=0$ outside S, but now

$$\phi' \to 0 \quad \text{as} \quad x^2+y^2+z^2 \to \infty\,. \tag{3.14}$$

Further, in the limit $\varepsilon \to 0$ the meter disappears altogether and we must have $\phi \equiv y$; thus $\phi' \to 0$ as $\varepsilon \to 0$. Substituting into (3.8) and considering only the leading terms as $\varepsilon \to 0$, we find

$$\phi'(x, 0, z) = \Gamma(x, z)\,. \tag{3.15}$$

The boundary value problem for ϕ' is thus a classical Dirichlet problem, values of ϕ' being prescribed on all boundaries. A separate physical interpretation of ϕ' is as the temperature distribution in the half-space $y \geqq 0$ produced by a hypothetical "hot spot" of temperature $\Gamma(x, z)$ over that part of the boundary $y=0$ occupied by the meter, the remainder of this plane being held at the same temperature ($\phi'=0$) as the ambient temperature at infinity. This problem for ϕ' is of course classical, and may be solved by a number of techniques, examples of which follow.

We mention in passing a slight advantage of the inconsistent, but somewhat more accurate, boundary condition (3.8). Meters are frequently constructed with shielding edges, i.e. portions of the meter have effectively zero conductivity, so that $\bar{K}=0$ and $\Gamma=\infty$. Clearly (3.15) breaks down in this case, whereas (3.8) is still valid, and correctly predicts that $\phi_y(x, 0, z)=0$ in the shielded regions. If indeed the regions of shielding are quite distinct we may still retain (3.15) in unshielded regions, but will then need to treat shielded regions separately. Uniform use of (3.8) obviates the need for this separate treatment, at the expense of a more difficult solution procedure, as we shall see.

4. Output Procedure and Design Philosophy

A heat flux meter is designed to attempt to measure the undisturbed heat flux in the surroundings, i.e. to estimate $\kappa T'$, using quantities measured by devices such as thermopiles situated in the meter. In general we may suppose that the available information consists of a distribution of temperature differences across the faces of the meter, or more directly, the actual heat flux through the meter.

From equation (3.5) we see that this measurable quantity must be $H(x, z)$ per unit external temperature gradient, which is obtained in terms of the outer field ϕ by (3.7). That is, (3.7) asserts that the quantity $\phi_y(x, 0, z)$ represents the "gain" of the meter, or the ratio between the measured (dimensional) heat flux $T'H(x, z)$ through the meter and the external heat flux $\kappa T'$. Since (3.13) implies that

$$\phi_y(x, 0, z) = 1 + \phi'_y(x, 0, z)\,, \tag{4.1}$$

we see that $\phi'_y(x, 0, z)$ measures the *error* in the meter at location $(x, 0, z)$, in the sense that if

$\phi'_y=0$ the meter is effectively transparent to heat at this location and the measured heat flux through the meter at this location is identical to that which would be present in the absence of the meter.

In practice we should expect the meter to record the average heat flux over some region, say A, of the (x, y) plane situated entirely within the meter. Here "A" may be the complete meter centre plane, or perhaps only a small fraction near its centre. Thus we define the gain

$$G = \frac{1}{A}\iint_A \phi_y(x, 0, z)dx\,dz = 1 + \frac{1}{A}\iint_A \phi'_y(x, 0, z)dx\,dz\,. \tag{4.2}$$

Since $\phi' = O(\varepsilon)$, it is clear that $G \to 1$ as $\varepsilon \to 0$. That is, an infinitesimally thin meter is always perfect. A gain of unity is also achieved, for any ε, by using a uniform meter conductivity $\bar{K}$ identical to that of the surroundings κ, for then the impedance $\Gamma=0$ and $\phi'=0$. However, neither of these possibilities is exactly achievable in practice.

In fact the real design problem is at fixed non-zero values of ε to choose a geometric shape function $F(x, z)$ and mean conductivity $\bar{K}(x, z)$ such that the gain G is affected as little as possible by changes in the external conductivity κ. This is achievable by making $G=1$ so long as $\bar{K}$ is constant. If $\bar{K}$ is not constant we can still estimate dependence on κ, leaving a meter which can be calibrated once and for all.

Considering first the simpler case $\bar{K}=$constant, we observe from (3.10) that the impedance Γ involves the constants ε, κ and $\bar{K}$ separately from the geometric shape function $F(x, z)$, and we can write

$$\phi'(x, y, z) = \left(\frac{\kappa}{\bar{K}} - 1\right)\varepsilon\phi''(x, y, z) \tag{4.3}$$

where $\phi''(x, y, z)$ is a canonical potential satisfying Laplace's equation in $y \geqq 0$, but on $y=0$ satisfies

$$\phi''(x, 0, z) = F(x, z)\,. \tag{4.4}$$

Thus ϕ'' depends *only* on geometry of the meter, and *not at all* on any of the constants ε, κ and $\bar{K}$.

Hence the gain G follows from (4.2) as

$$G = 1 + \alpha\left(1 - \frac{\kappa}{\bar{K}}\right)\varepsilon \tag{4.5}$$

where

$$\alpha = -\frac{1}{A}\iint_A \phi''_y(x, 0, z)dx\,dz \tag{4.6}$$

is a pure constant, depending only on geometry. For instance Philip [4] found $\alpha=\pi/2$ for meters in the forms of oblate spheroids. The formula (4.5) is an approximation of that suggested by Philip [4] for arbitrary ε, and expanded upon by Schwerdtfeger [6]. We shall compute the constant α for other special geometries later.

What does not appear to have been observed previously is that it is quite easy to design a meter such that α *vanishes* entirely. This means that, to the present order of approximation with respect to ε the meter is perfect, in that it possesses a gain of unity, and in particular is independent of κ. Equation (4.5) is of course only the first term in an asymptotic expansion with respect to ε, and by forcing $\alpha=0$ we have only cancelled the linear term in ε, leaving still possible contributions of $O(\varepsilon^2)$, $O(\varepsilon^3)$ etc. However, for any given ε, it would seem likely that $\alpha=0$ gives the best possible result.

In the more general situation in which $\bar{K}(x, z)$ is non-constant, we observe from (3.10) that we can write

$$\Gamma(x, z) = \varepsilon\kappa\left(\frac{F(x, z)}{\bar{K}(x, z)}\right) - \varepsilon F\,, \tag{4.7}$$

so that if we set

$$\phi'(x, y, z) = \varepsilon\kappa\phi'''(x, y, z) - \varepsilon\phi''(x, y, z)\,, \tag{4.8}$$

then ϕ'' still satisfies (4.4), but now

$$\phi'''(x, 0, z) = \frac{F(x, z)}{\bar{K}(x, z)}\,. \tag{4.9}$$

That is, the new canonical potential ϕ''' depends on geometry *and* on meter conductivity, but not on ε or κ. The gain of this meter is now representable as

$$G = 1 + \varepsilon(\alpha - \beta\kappa)\,, \tag{4.10}$$

where α is as defined in (4.6), whereas

$$\beta = -\frac{1}{A}\iint_A \phi_y'''(x, 0, z)\,dx\,dz\,. \tag{4.11}$$

If we are able to force *both* α and β to vanish simultaneously (e.g. by using variations in F alone to eliminate α, then variations in $\bar{K}$ alone at fixed F to eliminate β) we shall again have designed a unit gain meter. However, it is easier to simply force β to vanish, leaving a gain $G = 1 + \varepsilon\alpha$ which depends on thickness ε, but not at all on external conductivity κ.

Since the boundary value problems for ϕ', ϕ'', ϕ''' are very similar, in the following we can consider only ϕ'. Results for ϕ'', ϕ''' are obtained by substituting F, $F/\bar{K}$ respectively for Γ.

5. Two-Dimensional Meters

In the case in which there is no dependence on the z-coordinate, i.e. $\Gamma = \Gamma(x)$ only, the solution of the boundary value problem for ϕ' subject to (3.15) may be written down immediately in the form of a distribution of line dipoles, i.e.

$$\phi'(x, y) = \frac{1}{\pi}\frac{\partial}{\partial y}\int_{-1}^{1} \Gamma(\xi)\log\left[(x-\xi)^2 + y^2\right]^{\frac{1}{2}} d\xi\,. \tag{5.1}$$

This expression is readily seen to satisfy Laplace's equation, to vanish at infinity and on $y = 0$, $|x| > 1$, and to take the value $\Gamma(x)$ on $y = 0$, $|x| < 1$, as required.

Now we have

$$\phi_y'(x, y) = -\frac{1}{\pi}\frac{\partial^2}{\partial x^2}\int_{-1}^{1} \Gamma(\xi)\log\left[(x-\xi)^2 + y^2\right]^{\frac{1}{2}} d\xi \tag{5.2}$$

so that

$$\phi_y'(x, 0) = \frac{d}{dx}\mathscr{H}\Gamma(x) \tag{5.3}$$

where "$\mathscr{H}$" denotes the finite Hilbert transform operator (Tricomi, [8], p. 173), such that

$$\mathscr{H}\Gamma(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\Gamma(\xi)\,d\xi}{\xi - x}\,. \tag{5.4}$$

If we now average over the central portion $|x| < a < 1$ of the meter and assume the meter is symmetric about $x = 0$, we obtain from (4.2) a gain of

$$G = 1 + \frac{1}{a}\mathscr{H}\Gamma(a)\,. \tag{5.5}$$

The formula (5.5) may now be used for any choice of $\Gamma(x)$ and of a. For instance, with an "elliptical" meter of impedance

$$\Gamma(x) = \Gamma_0(1 - x^2)^{\frac{1}{2}} \tag{5.6}$$

where Γ_0 is some constant, we have

$$\mathscr{H}\Gamma(x) = -\Gamma_0 x\,, \tag{5.7}$$

so that

$$G = 1 - \Gamma_0\,. \tag{5.8}$$

Notice that in this case G is independent of a, and it is immaterial over what region of the meter we measure heat flux.

On the other hand, for a "rectangular" meter such that $\Gamma(x) = \Gamma_0 = \text{constant}$, we have

$$G = 1 - \frac{\Gamma_0}{a\pi} \log \frac{1+a}{1-a}\,. \tag{5.9}$$

Now not only does G depend on a, but we are unable to use the case $a = 1$, i.e. to measure over the whole width of the meter. This is an end-effect singularity and is no doubt due to the particular approximate theory being used for small ε; however, there may be some physical significance to this singularity and it may be a warning not to use values of a near to unity. In fact if we use a value of $a \ll 1$, i.e. only measure over a small region near the centre of the meter, then (5.9) reduces to

$$G = 1 - \frac{2}{\pi}\Gamma_0 \tag{5.10}$$

for rectangular meters, which is to be compared with (5.8) for elliptical meters.

In the case when $\bar{K}$ is constant, the constant Γ_0 appearing above may, from (3.10), be identified as

$$\Gamma_0 = \varepsilon\left(\frac{K}{\bar{K}} - 1\right), \tag{5.11}$$

so that (5.8) and (5.10) are special cases of (4.5) with $\alpha = 1$ and $\alpha = 2/\pi = 0.64$ respectively. These values of α are comparable with those suggested by Philip [4] as general estimates; however, we show later how it is easy to design meters which *any* value of α, including negative or zero values.

It is instructive at this point to consider also the problem of solving for the total field $\phi(x, y)$ subject to the mixed boundary condition (3.8). This is necessary in any case of shielded meters in which $\Gamma(x) = \infty$ for some range of values of x, and also, incidentally, avoids the end-effect singularity encountered above for the rectangular meter. If we again use a distribution of dipoles, we have

$$\phi(x, y) = y + \frac{1}{\pi}\frac{\partial}{\partial y}\int_{-1}^{1} \phi(\xi, 0) \log\left[(x-\xi)^2 + y^2\right]^{\frac{1}{2}} d\xi \tag{5.12}$$

as in (5.1), but where the quantity $\phi(\xi, 0)$ is now unknown. The boundary condition (3.8) implies (*cf.* (5.3)) that

$$\phi(x, 0) = \Gamma(x)\left[1 + \frac{d}{dx}\mathscr{H}\phi(x, 0)\right]. \tag{5.13}$$

In view of the integral definition of the $\mathscr{H}$ operator, equation (5.13) is an integro-differential equation, equivalent to the "airfoil equation" (Muskhelishvili, [3], p. 373). Many effective numerical procedures are available to solve this equation for given $\Gamma(x)$. We merely observe here that the *exact* solution of (5.13) for an elliptical meter with $\Gamma(x)$ given by (5.6) is

$$\phi(x, 0) = \frac{\Gamma_0}{1+\Gamma_0}(1-x^2)^{\frac{1}{2}} = \frac{\Gamma(x)}{1+\Gamma_0}\,. \tag{5.14}$$

The resulting expression for the gain is

$$G = \frac{1}{1+\Gamma_0}\,, \tag{5.15}$$

which agrees with (5.8) when Γ_0 is small, as it is of course, since ε is small. In fact, when (5.11) is substituted in (5.15) we obtain the exact solution for an elliptical meter of arbitrary thickness ε.

6. Axisymmetric Meters

Two-dimensional meters are only approximately realizable and are probably undesirable in any case. On the other hand, disc-like meters with the y-axis as an axis of symmetry are perfectly practical, and actual meters are generally of this nature. We therefore assume now that all potentials ϕ, ϕ', ϕ'', ϕ''' depend on the coordinates r, y only, where

$$r=(x^2+z^2)^{\frac{1}{2}}\,, \tag{6.1}$$

and that Γ is a function of r only.

Although the axisymmetric problem bears a strong analogy with the two-dimensional problem, the analysis is much more difficult, and we give only an outline of two possible approaches here. A "solution" such as that given in (5.1) is easy enough to write down formally, but difficult to work with. Thwaites ([7], p. 379) gives a formula for the potential of a ring source, from which we can obtain a ring dipole by y-differentiation.

Thus the solution for $\phi'(r, y)$ subject to (3.15) is

$$\phi'(r, y) = -\frac{2}{\pi}\frac{\partial}{\partial y}\int_0^1 \Gamma(\rho)\{\mathsf{K}(m)/R\}d\rho\,, \tag{6.2}$$

where

$$R^2=(r+\rho)^2+y^2\,, \tag{6.3}$$

$$m^2 = 4\rho r/R^2\,, \tag{6.4}$$

and

$$\mathsf{K}(m) = \int_0^{\pi/2} (1-m^2\sin^2\theta)^{-\frac{1}{2}}d\theta \tag{6.5}$$

is a complete elliptic integral (Abramowitz and Stegun [1], p. 590). Analogous to (5.3) we have

$$\phi_y'(r, 0) = \frac{1}{r}\frac{d}{dr}\,r\mathscr{E}\Gamma(r) \tag{6.6}$$

where

$$\mathscr{E}\Gamma(r) = \frac{2}{\pi}\frac{d}{dr}\int_0^1 \Gamma(\rho)\,\frac{\mathsf{K}\left\{\left(1-\left(\frac{r-\rho}{r+\rho}\right)^2\right)^{\frac{1}{2}}\right\}}{r+\rho}d\rho\,. \tag{6.7}$$

The operator $\mathscr{E}$ defined by (6.7) is analogous to the Hilbert transform operator $\mathscr{H}$; in particular, its singular character as $\rho\to r$ is the same, in view of the logarithmic behaviour of $\mathsf{K}(m)$ as $m\to 1$ (Abramowitz and Stegun, [1], p. 591). However, $\mathscr{E}$ is clearly much more difficult to work with than $\mathscr{H}$, and we seek an alternative simpler approach.

The axisymmetric problem may also be solved using oblate spheroidal coordinates (μ, ζ) defined by

$$r=(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}\,,\quad y=\mu\zeta\,. \tag{6.8}$$

We seek a solution of the form (Lamb, [2], p. 143)

$$\phi' = \sum_{m=0}^{\infty} A_m P_{2m+1}(\mu)\,\frac{Q_{2m+1}(i\zeta)}{Q_{2m+1}(i0)}\,, \tag{6.9}$$

where P_{2m+1}, Q_{2m+1} are Legendre functions of odd order (Abramowitz and Stegun [1], p. 332) and A_m are coefficients to be determined. We observe that on $y=0$ *outside* the meter we have $\mu=0$ and (6.9) satisfies $\phi'=0$ as required, whereas on $y=0$ *inside* the meter we have $\zeta=0$ and

$$\phi' = \sum_{m=0}^{\infty} A_m P_{2m+1}(\mu) \tag{6.10}$$

where now (6.8) reduces to $r=(1-\mu^2)^{\frac{1}{2}}$.

Thus, using (3.15), the problem has reduced to that of finding the Fourier–Legendre coefficients A_m of $\phi'(r, 0)=\Gamma$, considered as an *odd* function of $\mu=(1-r^2)^{\frac{1}{2}}$. These coefficients are determined by orthogonality, thus

$$A_m = \frac{4m+3}{2} \int_{-1}^{1} \Gamma(\mu) P_{2m+1}(\mu) d\mu \,. \tag{6.11}$$

Once the coefficients A_m are known, the gain follows by computation of

$$\phi'_y(r, 0) = \frac{1}{\mu} \frac{\partial}{\partial \zeta} \phi' \Big|_{\zeta=0} .$$

Thus

$$\phi'_y(r, 0) = \sum_{m=0}^{\infty} A_m q_m \frac{P_{2m+1}(\mu)}{\mu} \tag{6.12}$$

where

$$q_m = \frac{\mathrm{i} Q'_{2m+1}(\mathrm{i}0)}{Q_{2m+1}(\mathrm{i}0)} \tag{6.13}$$

$$= -\frac{\pi}{2} \frac{((2m+1)!)^2}{2^{4m}(m!)^4} . \tag{6.14}$$

Equation (6.14) follows from (6.13) by first deriving the recurrence relation

$$q_m = \left(\frac{2m+1}{2m}\right)^2 q_{m-1} \tag{6.15}$$

and observing that $q_0 = -\pi/2$.

Averaging over the circular centre region $0 \leq r < a < 1$, we have

$$G = 1 + \sum_{m=0}^{\infty} A_m q_m \frac{2}{a^2} \int_{(1-a^2)^{1/2}}^{1} P_{2m+1}(\mu) d\mu \,. \tag{6.16}$$

If $a \ll 1$, (6.16) reduces to

$$G = 1 + \sum_{m=0}^{\infty} A_m q_m = 1 - \frac{\pi}{2} \sum_{m=0}^{\infty} \left(\frac{(2m+1)!}{2^{2m} m!^2}\right)^2 A_m \,. \tag{6.17}$$

Although the above analysis is obviously extremely complicated, the final result (6.17) is reasonably simple and provides a direct answer for a given Γ, once the integrals (6.11) have been evaluated to give A_m. For example we may consider an "oblate spheroidal" meter with impedance

$$\Gamma = \Gamma_0 (1-r^2)^{\frac{1}{2}} = \Gamma_0 \mu = \Gamma_0 P_1(\mu) \,. \tag{6.18}$$

This is a particularly simple case, for it is clear from independence of Legendre polynomials that

$$A_0 = \Gamma_0 \,, \quad A_m = 0 \,, \qquad m = 1, 2, 3, \ldots \,. \tag{6.19}$$

Thus from (6.17),

$$G = 1 - \frac{\pi}{2} \Gamma_0 \,, \tag{6.10}$$

giving an α-value of $\pi/2$ in equation (4.5). In fact this is of course the exact solution as given by Philip [4], and does not require $a \ll 1$ for its validity.

Somewhat more generally, we may consider the class of meters for which A_0 and A_1 are non-zero, but all other A_m vanish. Then

$$\Gamma = A_0\mu + \frac{A_1}{2}(5\mu^3 - 3\mu) = \Gamma_0(1-r^2)^{\frac{1}{2}}(1+\gamma r^2) \tag{6.21}$$

where

$$\Gamma_0 = A_0 + A_1 \tag{6.22}$$

is the value of Γ at $r=0$, and

$$\gamma = -5A_1/2\Gamma_0 . \tag{6.23}$$

The corresponding gain is

$$G = 1 - \frac{\pi}{2}(A_0 + \tfrac{9}{4}A_1) = 1 - \frac{\pi}{2}\Gamma_0(1-\tfrac{1}{2}\gamma) . \tag{6.24}$$

Thus if the meter is uniform and the thickness ratio ε is based on thickness at the centre, so that (5.11) applies, then (4.5) gives

$$\alpha = \frac{\pi}{2}(1-\tfrac{1}{2}\gamma) . \tag{6.25}$$

Thus, as γ varies, we obtain from (6.21) a family of meter shapes (since Γ is proportional to F for constant $\bar{K}$) with varying values of the gain coefficient α given by (6.25). Some examples of these shapes are shown in Figure 3. Note that $\gamma > -1$ is a necessary condition. Shapes with $\gamma > \frac{1}{2}$ are concave, such that $\Gamma > \Gamma_0$ for some values of r. If $\bar{K}$ is not constant we can also achieve the required values of Γ in (6.21) by *inverse* variation of $\bar{K}(r)$ rather than by shape variations.

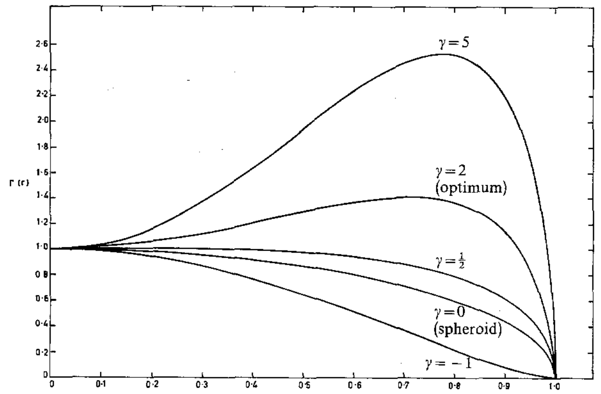

Figure 3. A family of axisymmetric meters.

By using families with more and more non-zero coefficients A_m we can build up close approximations to any given shape. For example, the case $\gamma = \frac{1}{2}$ above gives the nearest convex approximation to a rectangular disc meter with Γ = constant among the two-coefficient family, with $\alpha = 3\pi/8 = 1.17$. More nearly rectangular shapes may be constructed by using more coefficients; however, it must be realized that these families all have the property that $\Gamma = 0$ at the edge $r=1$, so that we can never exactly achieve a sharp "edge" on the disc.

The mixed problem (3.8) may also be attacked for axisymmetric meters, and an integro-differential equation analogous to the airfoil equation (5.13) results, but now involving the integral operator $\mathscr{E}$ instead of $\mathscr{H}$. Clearly this problem is much more difficult to handle. However, we observe in passing that again the oblate spheroidal meter leads to a simple exact solution, as for the elliptical meter.

7. Ideal and Optimum Meters

We may define an "ideal" meter as one in which

$$\phi'_y(x, 0, z) = 0 \tag{7.1}$$

everywhere inside the meter. Such a meter, if it could be constructed, would not disturb the ambient field at all in the neighbourhood of its centre plane, and the gain G would be unity for all possible averaging regions A. It is instructive to consider the problem of designing such a meter, and we use the two-dimensional case to illustrate this problem.

Indeed, the only possible (even) choice for $\Gamma(x)$ such that $\phi'_y(x, 0)$ in (5.3) vanishes is (Tricomi, [8], p. 174)

$$\Gamma(x) = \frac{\Gamma_0}{(1-x^2)^{\frac{1}{2}}} \tag{7.2}$$

for some constant Γ_0. That is, we must force the impedance $\Gamma(x)$ to have a square root singularity at the edges $x = \pm 1$ of the meter. Recalling from (3.10) the definition of Γ, we observe that either the edge thickness F must be infinite or else the edge conductivity $\bar{K}$ must be zero. The former is clearly impossible, while the latter may only be approximated by thin insulating shields.

Equation (7.2) is useful, nevertheless, in indicating the general form of the ideal impedance, namely low (e.g. thin) at the centre and high (e.g. thick) at the edges. One way of eliminating an actual edge singularity is to provide a finite width insulating shield, inside of which we again force $\phi'_y = 0$. The solution of this problem is given in the Appendix, and the resulting "ideal shielded meter" shapes are shown in Figure 5.

In order to obtain a result analogous to (7.2) for the axisymmetric case, we should need to solve the equation

$$r\mathscr{E}\Gamma(r) = \text{constant}, \tag{7.3}$$

which follows from (6.6) with $\phi'_y = 0$. Although this has not yet been done, we can infer from the general theory of singular integral equations (Muskhelishvili [3], p. 249) that the resulting $\Gamma(r)$ will have an inverse square root singularity at $r = 1$, i.e. that the general shape of the ideal meter will be quite like (7.2).

Fortunately, it is not necessary that a meter be "ideal" in order that a gain of unity be achievable for a *particular* choice of the averaging region A, even if not for every possible A. For instance, consider a two-dimensional uniform-conductivity meter consisting of a thin centre strip with thick edges, thus

$$\Gamma(x) = \begin{cases} \Gamma_1, & |x| < b \\ \Gamma_2, & b < |x| < 1, \end{cases} \tag{7.4}$$

where $\Gamma_1 < \Gamma_2$. We find

$$\mathscr{H}\Gamma(x) = -\frac{\Gamma_2}{\pi} \log \frac{1+x}{1-x} + \frac{\Gamma_2 - \Gamma_1}{\pi} \log \frac{b+x}{b-x}. \tag{7.5}$$

If $a \ll b$, the gain from a measuring region $|x| < a$ is (equation (5.5))

$$G = 1 - \frac{2}{\pi}\Gamma_2 + \frac{2}{\pi b}(\Gamma_2 - \Gamma_1) \tag{7.6}$$

which takes the value unity if

$$\Gamma_1 = (1-b)\Gamma_2. \tag{7.7}$$

The corresponding result for general (non-small) values of a is

$$\Gamma_1 = \left[1 - \frac{\log \frac{1+a}{1-a}}{\log \frac{b+a}{b-a}}\right]\Gamma_2. \tag{7.8}$$

If the meter is indeed uniform in conductivity, equation (7.7) gives the relationship between the thicknesses and shows that the centre portion $|x|<b$ should be thinner than the outside portion, in the ratio $(1-b):1$. On the other hand, we may prefer to use a constant thickness, and to adjust the impedance Γ by varying the conductivity $\bar{K}$ in (3.10). In that case we cannot expect to achieve a gain of unity, but now use the boundary condition (4.9) for ϕ''', effectively replacing Γ by $F/\bar{K}$. Then for uniform thickness $F=$constant, (7.7) gives the required inverse conductivity, and says that the conductivity of the centre portion should be higher than that of the edges in the ratio $1:(1-b)$ in order that the parameter β in (4.10) should vanish.

Similar analysis may be carried out for the axisymmetric case. In fact, equation (6.25) already gives us the result that the optimum member of the family (6.21) has $\gamma=2$. This optimum shape is one of those shown in Figure 3; the impedance rises to a maximum of about 1.4 Γ_0 at $r=0.7$ before returning to its required zero value at the edge $r=1$. This is of course only one example of an optimum distribution of impedance; examples of more practical significance may be constructed using more coefficients A_m.

Acknowledgement

I wish to thank Prof. P. Schwerdtfeger for introducing me to this problem, and Dr. J. R. Philip for a number of helpful suggestions.

Appendix

Ideal Shielded Meters

Consider the two-dimensional problem for ϕ' illustrated by Figure 4. The centre portion $|x|<b$ is one on which we force the heat flux *disturbance* ϕ'_y to vanish in order to obtain an ideal meter, while the "shields" $b<x<1$ and $-1<x<-b$ are regions in which the *total* heat flux $\phi_y=1+\phi'_y$ vanishes.

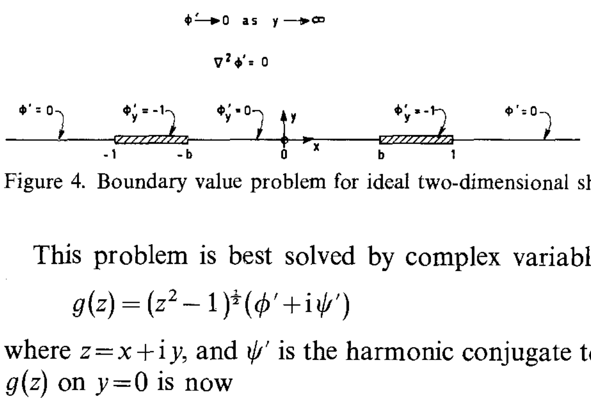

Figure 4. Boundary value problem for ideal two-dimensional shielded meters.

This problem is best solved by complex variable methods, defining a complex potential

$$g(z)=(z^2-1)^{\frac{1}{2}}(\phi'+i\psi') \tag{A.1}$$

where $z=x+iy$, and ψ' is the harmonic conjugate to ϕ'. The boundary condition satisfied by $g(z)$ on $y=0$ is now

$$\mathrm{Re}\, g(z)=\begin{cases} 0, & |x|>1 \text{ and } |x|<b \\ (b-x)(1-x^2)^{\frac{1}{2}}, & b<x<1 \\ (-b-x)(1-x^2)^{\frac{1}{2}}, & -1<x<-b\,. \end{cases} \tag{A.2}$$

Since Re $g(z)\to 0$ at infinity, Cauchy's integral theorem gives on $y=0$

$$\mathrm{Im}\, g(x)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\, g(\xi)}{\xi-x}\,d\xi+\Gamma_0\,, \tag{A.3}$$

where Γ_0 is an arbitrary constant. Thus on use of (A.2) we obtain

$$\mathrm{Im}\, g(x)=\Gamma_0+\frac{2}{\pi}\int_b^1\frac{\xi(\xi-b)(1-\xi^2)^{\frac{1}{2}}}{\xi^2-x^2}\,d\xi\,. \tag{A.4}$$

For $|x| < 1$, $\phi' = (1-x^2)^{-\frac{1}{2}} \operatorname{Im} g(x)$, so that finally

$$\phi'(x,0) = \frac{\Gamma_0}{(1-x^2)^{\frac{1}{2}}} + \frac{2/\pi}{(1-x^2)^{\frac{1}{2}}} \int_b^1 \frac{\xi(\xi-b)(1-\xi^2)^{\frac{1}{2}}}{\xi^2 - x^2} d\xi \qquad \text{(A.5)}$$
$$= \Gamma(x) \text{ by (3.15)}.$$

As $b \to 1$ the shields disappear and we recover the result (7.2), since the integral in (A.5) vanishes. For $b \neq 1$, the formula (A.5) leads, in contrast to (7.2), to a finite value of Γ for $|x| \leqq b$, and in particular,

$$\Gamma(b) = \frac{\Gamma_0}{(1-b^2)^{\frac{1}{2}}} + \frac{2/\pi}{(1-b^2)^{\frac{1}{2}}} \int_b^1 \frac{\xi(1-\xi^2)^{\frac{1}{2}}}{\xi+b} d\xi \qquad \text{(A.6)}$$

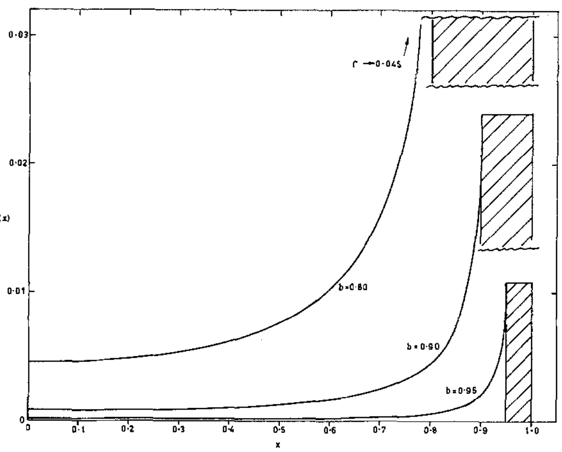

Figure 5. Ideal two-dimensional shielded meters with $\Gamma_0 = 0$.

Figure 5 shows the contribution from the integral in (A.5) alone, i.e. the design with $\Gamma_0 = 0$, for several values of b. For b very close to unity, the impedance required is very small near the centre; this may be corrected by using a suitably large value of Γ_0.

REFERENCES

[1] M. Abramowitz and L. A. Stegun (eds.), Handbook of Mathematical Functions. National Bureau of Standards, *Applied Mathematics Series* No. 55, Washington, D.C. (1964).

[2] H. Lamb, *Hydrodynamics* (6th edn.) Cambridge and Dover (1932).

[3] N. I. Muskhelishvili, *Singular Integral Equations* (Translated by J. R. M. Radok) Groningen, Holland, Noordhoff (1963).

[4] J. R. Philip, The theory of heat flux meters. *J. Geophys. Res.* 66 (1961) 571–579.

[5] D. J. Portman, Conductivity and length relationships in heat-flow transducer performance, *Trans. Am. Geophys. Union*, 39 (1958) 1089–1094.

[6] P. Schwerdtfeger, *The measurement of heat flow in the ground and the theory of heat flux meters.* Cold Regions Research and Engineering Lab. Report, Hanover, New Hampshire, November (1970).

[7] B. Thwaites (ed.), *Incompressible Aerodynamics*, Oxford (1960).

[8] F. G. Tricomi, *Integral Equations*, Interscience (1957).

[9] M. Van Dyke, *Perturbation methods in fluid mechanics.* Academic Press (1964).